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Explicit determination of a 727-dimensional root space of the hyperbolic Lie algebra E_{10}

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Abstract. The 727-dimensional root space associated with the level-2 root $\mathbf{\Lambda}_1$ of the hyperbolic Kac–Moody algebra E_{10} is determined using a recently developed string theoretic approach to hyperbolic algebras. The explicit form of the basis reveals a complicated structure with transversal as well as longitudinal string states present.

1. Introduction

Finite and affine Lie algebras represent well-understood classes of Kac–Moody algebras (see, e.g., [1, 2]). The Kac–Moody algebras associated with indefinite Cartan matrices, however, have so far resisted attempts at systematic and conceptual understanding. The difficulties arise from the very definition of these Lie algebras in terms of generators and relations. This definition entails that we have no basis for indefinite Kac–Moody algebras and so must form arbitrary multiple commutators of the basic Chevalley generators and must take into account the Jacobi identity and the complicated Serre relations. Yet this approach is intractible from a practical point of view since it would be extremely cumbersome to actually *write down* Lie algebra elements as multiple commutators. In fact, there is not a single example of an indefinite Kac–Moody algebra for which all root multiplicities, let alone an explicit basis, are known. Therefore the crucial first step towards a thorough analysis of these infinite-dimensional Lie algebras is the search for other realizations of them (which would be analogous to the realization of affine Lie algebras as central extensions of loop algebras).

Surprisingly, although perhaps not unexpectedly, Lorentzian Kac–Moody algebras, which are indefinite Kac–Moody algebras with Lorentzian signature of the associated root lattice, naturally appear in theoretical physics, namely, as subalgebras of the Lie algebra of physical states of a completely compactified chiral bosonic string with the Lorentzian root lattice as momentum lattice (cf [3–5]). It turns out that for the uncompactified string a complete basis for the space of physical states is provided by the so-called DDF construction [6, 7]. So we do have explicit realizations of the bigger Lie algebra of physical states and the problem is now to single out those DDF states which lead to a basis of the embedded Lorentzian Kac–Moody algebra.

In [8], a discrete version of the DDF construction was developed and exploited to tackle the above problem for the case of the hyperbolic Kac–Moody algebra E_{10}. It turned out that the level-1 sector (which is known to be isomorphic to the basic representation of the affine subalgebra) is spanned by the transversal DDF states whereas the longitudinal states are all 'missing' in the sense that they lie in the orthogonal complement of the hyperbolic algebra within the full Lie algebra of physical states. As a demonstration of the utility of this new approach, the root space of the fundamental level-2 root $\mathbf{\Lambda}_7$ was analysed and a 192-dimensional DDF basis was found. The latter involved not only transversal states but also string states with longitudinal excitations.

In this paper, we apply the DDF analysis to the fundamental level-2 root $\mathbf{\Lambda}_1$ and exhibit a 727-dimensional basis for the associated root space of the hyperbolic algebra E_{10}. From the complicated structure in terms of transversal and longitudinal polarization tensors (see page 9), we draw the following conclusions. First, it is clear that the longitudinal DDF operators (which in general form a Virasoro algebra with central charge $26-d$) will play an essential role in understanding the higher-level sectors of the hyperbolic algebra (cf [9, 10]). Second, hyperbolic Kac–Moody algebras seem to have some additional hidden structure which cannot be revealed by the perturbative string model alone (see, e.g., [11] for further speculations).

Let us briefly describe how the paper is organized. After introducing the string model in the framework of vertex algebras, we review the discrete DDF construction for the example E_{10}. Finally, we present the explicit basis for the analysed root space. The necessary calculations are collected in an appendix. For further details we refer the reader to the diploma thesis [12], on which the present paper is based.

2. Vertex algebras and compactified strings

We shall study one chiral sector of a closed bosonic string moving on a Minkowski torus as spacetime, i.e. with all target space coordinates compactified. Uniqueness of the quantum mechanical wavefunction then forces the centre of mass momenta of the string to form a lattice with Minkowskian signature.

Upon 'old' covariant quantization this system turns out to realize a mathematical structure called vertex algebra [5]. For a detailed account of this topic the reader may wish to consult the review [13]. Here we will follow closely [8], omitting most of the technical details.

We work with *formal variables* $z_1, z_2, \ldots$ and formal Laurent series, an algebraic approach to complex analysis. For a vector space S, we define the $\mathbb{C}$-vector space of formal series as $S[\![z, z^{-1}]\!] = \{\sum_{n\in\mathbb{Z}} s_n z^n | s_n \in S\}$. For a formal series we use the residue notation

$$\mathrm{Res}_z\left[\sum_{n\in\mathbb{Z}} s_n z^n\right] = s_{-1}. \tag{1}$$

A thorough introduction to formal calculus can be found in [14]. In the string model we shall consider the formal variables as having their origin as complex worldsheet coordinates.

A *vertex algebra* $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \boldsymbol{\omega})$ is a $\mathbb{Z}$-graded vector pace $\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}^n$, equipped with an injective linear map $\mathcal{V} : \mathcal{F} \mapsto (\mathrm{End}\mathcal{F})[\![z, z^{-1}]\!]$, which assigns to every *state* $\psi \in \mathcal{F}$ a *vertex operator* $\mathcal{V}(\psi, z)$. As operator-valued formal Laurent series, vertex operators are completely determined by their mode operators defined by

$$\mathcal{V}(\psi, z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n-1} \tag{2}$$

where $\psi_n \in \mathrm{End}\mathcal{F}$ for all n. The state space $\mathcal{F}$ of a vertex algebra contains two preferred elements: the *vacuum 1* and the *conformal vector* $\boldsymbol{\omega}$ whose associated vertex operators are given by the identity $\mathrm{id}_{\mathcal{F}}$ and the stress–energy tensor

$$\mathcal{V}(\boldsymbol{\omega}, z) = \sum_{n\in\mathbb{Z}} \mathrm{L}_{(n)} z^{-n-2} \tag{3}$$

respectively. The latter provides the generators L_n of the *Virasoro algebra* Vir such that the grading of $\mathcal{F}$ is obtained by the eigenvalues of L_0 and the role of a translation generator is played by $\mathrm{L}_{(-1)}$ satisfying

$$\mathcal{V}(\mathrm{L}_{(-1)}\psi) = \frac{\mathrm{d}}{\mathrm{d}z}\mathcal{V}(\psi, z).$$

Finally, there is a crucial identity relating products and iterates of vertex operators called the *(Cauchy–) Jacobi identity*:

$$\sum_{i\geqslant 0}(-1)^i \binom{l}{i} (\psi_{l+m-i}\phi_{n+i} - (-1)^l \phi_{l+n-i}\psi_{m+i}) = \sum_{i\geqslant 0} \binom{m}{i} (\psi_{l+i}\phi)_{m+n-i} \tag{4}$$

for all $\psi, \phi \in \mathcal{F}, l, m, n \in \mathbb{Z}$.

An important property of vertex algebras is the *skew symmetry*

$$\mathcal{V}(\psi, z)\varphi = \mathrm{e}^{z\mathrm{L}_{(-1)}}\mathcal{V}(\varphi, -z)\psi \tag{5}$$

which shows that the vertex operator $\mathcal{V}(\psi, z)$ 'creates' the state $\psi \in \mathcal{F}$ from the vacuum, viz

$$\mathcal{V}(\psi, z)\mathbf{1} = \mathrm{e}^{z\mathrm{L}_{(-1)}}\psi. \tag{6}$$

In string theory a special role is played by the subspace $\mathcal{P}^1 \subset \mathcal{F}^1$, the space of *(conformal) highest weight vectors* or *primary states* of weight one, satisfying

$$\mathrm{L}_{(0)}\psi = \psi \tag{7a}$$

$$\mathrm{L}_{(n)}\psi = 0 \qquad \forall n > 0. \tag{7b}$$

We shall call the states in $\mathcal{P}^1$ *physical states* from now on. The vertex operators associated with physical states enjoy rather simple commutation relations with the generators of Vir; in terms of the mode operators we have

$$[\mathrm{L}_{(n)}, \psi_m] = -m\psi_{m+n} \tag{8}$$

so that in particular the zero modes of physical states commute with the Virasoro constraints.

The quotient space $\mathcal{F}/\mathrm{L}_{(-1)}\mathcal{F}$ carries the structure of a Lie algebra with bracket defined by [5]

$$[\psi, \varphi] := \mathrm{Res}_z[\mathcal{V}(\psi, z)\varphi] = \psi_0\varphi \tag{9}$$

where the antisymmetry and the Jacobi identity follow from the skew symmetry (5) and the Cauchy–Jacobi identity (4), respectively. The Lie algebra $\mathcal{F}/\mathrm{L}_{(-1)}\mathcal{F}$ is too large for a further analysis. We will therefore concentrate on a subalgebra which is also relevant for string theory, namely the *Lie algebra of physical states*:

$$\mathfrak{g}_{\mathcal{F}} := \mathcal{P}^1/\mathrm{L}_{(-1)}\mathcal{P}^0. \tag{10}$$

In general, the physics described by vertex algebras corresponds to meromorphic bosonic two-dimensional conformal quantum field theories (see, e.g., [15]). Here, we will consider the vertex algebra associated with one chiral sector of a first quantized bosonic string theory moving on a Minkowski torus as target space. We will briefly review the main ingredients of the model, the details of which can be found in [8].

Let Λ be an even non-generate lattice of rank $d < \infty$, representing the lattice of allowed centre-of-mass momenta for the string. To each lattice point we assign a *zero mode state* $\mathrm{e}^{\boldsymbol{r}}$, which we alternatively denote by $|\boldsymbol{r}\rangle$ and which provides a highest weight vector for a d-fold Heisenberg algebra $\hat{\boldsymbol{h}}$ of oscillators $\alpha_m^\mu, n \in \mathbb{Z}, 1 \leqslant \mu \leqslant d$,

$$\alpha_0^\mu |\boldsymbol{r}\rangle = r^\mu |\boldsymbol{r}\rangle \qquad \alpha_m^\mu |\boldsymbol{r}\rangle = 0 \qquad \forall n > 0 \tag{11}$$

where

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}. \tag{12}$$

These zero mode states realize the ϵ-twisted group algebra $\mathbb{R}\{\Lambda\}$ for some 2-cocycle ϵ. For notational convenience we put $\boldsymbol{r}(m) \equiv \boldsymbol{r} \cdot \boldsymbol{\alpha}_m$ and $\boldsymbol{r}(z) \equiv \sum_{m\in\mathbb{Z}} \boldsymbol{r}(m) z^{-m-1}$. The Fock space is obtained by collecting the Heisenberg modules built on all zero mode states, viz

$$\mathcal{F} := S(\hat{\boldsymbol{h}}^-) \otimes \mathbb{R}\{\Lambda\}. \tag{13}$$

We now assign to each state $\psi \in \mathcal{F}$ a *vertex operator* $\mathcal{V}(\psi, z)$. For a zero mode state $\mathrm{e}^{\boldsymbol{r}}$ we define

$$\mathcal{V}(\mathrm{e}^{\boldsymbol{r}}, z) := \exp\left(\int \boldsymbol{r}_<(z)\,\mathrm{d}z\right) \mathrm{e}^{\boldsymbol{r}} z^{\boldsymbol{r}(0)} \exp\left(\int \boldsymbol{r}_>(z)\,\mathrm{d}z\right) \tag{14}$$

with $\boldsymbol{r}_<(z) = \sum_{m\in\mathbb{N}} \boldsymbol{r}(-m) z^{m-1}$ and $\boldsymbol{r}_>(z) = \sum_{m\in\mathbb{N}} \boldsymbol{r}(m) z^{-m-1}$. And for $\psi = \prod_{j=1}^N \boldsymbol{s}_j(-n_j) \otimes \mathrm{e}^{\boldsymbol{r}}$, a general homogeneous element of $\mathcal{F}$, we have

$$\mathcal{V}(\psi, z) :=: \mathcal{V}(\mathrm{e}^{\boldsymbol{r}}, z) \prod_{j=1}^N \frac{1}{(n_j-1)!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n_j-1} \boldsymbol{s}_j(z):. \tag{15}$$

This definition can be extended by linearity to the whole of $\mathcal{F}$. These operators indeed fulfil the axioms of a vertex algebra [14].

A special role is played by the lattice vectors of length two which are called the *real roots* of the lattice. We associate with every real root a *reflection* by $w_{\boldsymbol{r}}(\boldsymbol{x}) = \boldsymbol{x} - (\boldsymbol{x} \cdot \boldsymbol{r})\boldsymbol{r}$ for $\boldsymbol{x} \in \Lambda$. The hyperplanes perpendicular to these divide the real vector space $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda$ into regions called *Weyl chambers*. The reflections in the real roots of Λ generate a group called the *Weyl group* $\mathcal{W}$ of Λ, which acts simply transitively on the Weyl chambers. Fixing one chamber to be the *fundamental Weyl chamber* C once and for all, we call the real roots perpendicular to the faces of C and with inner product at most 0 with elements of C, the *simple roots* $\boldsymbol{r}_i$ of C.

The physical states

$$e_i := \mathrm{e}^{\boldsymbol{r}_i} \qquad f_i := -\mathrm{e}^{-\boldsymbol{r}_i} \qquad h_i := \boldsymbol{r}_i(-1)|\boldsymbol{0}\rangle \tag{16}$$

for all i then obey the following commutation relations (see [5]):

$$[h_i, h_j] = 0 \tag{17}$$

$$[h_i, e_j] = A_{ij} e_j \qquad [h_i, f_j] = -A_{ij} f_j \tag{18}$$

$$[e_i, f_j] = \delta_{ij} h_i \tag{19}$$

$$(\mathrm{ad}\, e_i)^{1-A_{ij}} e_j = 0 \qquad (\mathrm{ad}\, f_i)^{1-A_{ij}} f_j = 0\,(i \neq j) \tag{20}$$

i.e. they generate via multiple commutators the *Kac–Moody algebra* $\mathfrak{g}(A)$ associated with the Cartan matrix $A = (A_{ij}) = (\boldsymbol{r}_i \cdot \boldsymbol{r}_j)$, which is a subalgebra of the Lie algebra of physical states $\mathfrak{g}_\Lambda$. In the Euclidean case both algebras coincide but in general we have a proper inclusion

$$\mathfrak{g}(A) \subset \mathfrak{g}_\Lambda. \tag{21}$$

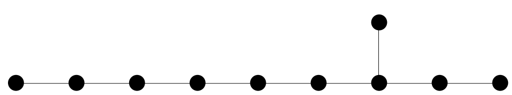
The main problem in this string realization of the hyperbolic Lie algebra is to determine the elements of $\mathfrak{g}_\Lambda$ *not* contained in $\mathfrak{g}(A)$, which we call *missing* or *decoupled states*.

3. E_{10} and the DDF construction

Most of the information about the hyperbolic Kac–Moody algebra E_{10} available in the mathematical literature can be found in [16]; readers interested in more general information about infinite-dimensional Lie algebras should consult the textbooks [1] or [2]. The hyperbolic Lie algebra E_{10} is defined via its Coxeter–Dynkin diagram and the Serre relations following from it. The root lattice $Q(E_{10})$ coincides with the unique 10-dimensional even unimodular Lorentzian lattice $II_{9,1}$. The latter can be defined as the lattice of all points $\boldsymbol{x} = (x_1, \ldots, x_9|x_0)$ for which the x_ms are all in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$ and which have integer inner product with the vector $\boldsymbol{l} = (\frac{1}{2}, \ldots, \frac{1}{2}|\frac{1}{2})$, all norms and inner products being evaluated in the Minkowskian metric $\boldsymbol{x}^2 = x_1^2 + \cdots + x_9^2 - x_0^2$. A basis of simple roots for this lattice is given by the 10 lattice vectors

$$
\begin{aligned}
\boldsymbol{r}_{-1} &= (0, 0, 0, 0, 0, 0, 0, 1, -1, 0)\\
\boldsymbol{r}_0 &= (0, 0, 0, 0, 0, 0, 1, -1, 0, 0)\\
\boldsymbol{r}_1 &= (0, 0, 0, 0, 0, 1, -1, 0, 0, 0)\\
\boldsymbol{r}_2 &= (0, 0, 0, 0, 1, -1, 0, 0, 0, 0)\\
\boldsymbol{r}_3 &= (0, 0, 0, 1, -1, 0, 0, 0, 0, 0)\\
\boldsymbol{r}_4 &= (0, 0, 1, -1, 0, 0, 0, 0, 0, 0)\\
\boldsymbol{r}_5 &= (0, 1, -1, 0, 0, 0, 0, 0, 0, 0)\\
\boldsymbol{r}_6 &= (-1, -1, 0, 0, 0, 0, 0, 0, 0, 0)\\
\boldsymbol{r}_7 &= (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})\\
\boldsymbol{r}_8 &= 1, -1, 0, 0, 0, 0, 0, 0, 0, 0).
\end{aligned}
$$

These simple roots indeed generate the reflection group of $II_{9,1}$. The corresponding Coxeter–Dynkin diagram appears as follows



The Cartan matrix is $A_{ij} = \boldsymbol{r}_i \cdot \boldsymbol{r}_j$. The fundamental Weyl chamber C of E_{10} is the convex cone generated by the *fundamental weights* $\boldsymbol{\Lambda}_i$ which obey $\boldsymbol{\Lambda}_i \cdot \boldsymbol{r}_j = -\delta_{ij}$ and are explicitly given by

$$
\boldsymbol{\Lambda}_i = -\sum_{j=-1}^{8} (A^{-1})_{ij} \boldsymbol{r}_j \qquad \text{for } i = -1, 0, 1, \ldots, 8 \tag{22}
$$

in terms of the inverse Cartan matrix. In the following, the E_9 null root $\boldsymbol{\delta}$, dual to the overextended simple root $\boldsymbol{r}_{-1}$, will play an important role; it is

$$
\boldsymbol{\delta} = \sum_{i=0}^{8} n_i \boldsymbol{r}_i = \left[\begin{array}{ccccccccc} & & & & & & 3 & & \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 \end{array} \right] = (0, 0, 0, 0, 0, 0, 0, 0, 1|1). \tag{23}
$$

A useful notion in the theory of hyperbolic Kac–Moody algebras is the *level* ℓ of a given root $\boldsymbol{\Lambda}$, which is defined as the number of times the simple root $\boldsymbol{r}_{-1}$ occurs in it. The formula for ℓ is

$$
\ell \equiv \ell(\boldsymbol{\Lambda}) := -\boldsymbol{\delta} \cdot \boldsymbol{\Lambda} \tag{24}
$$

where $\boldsymbol{\delta}$ denotes the null root given by (23). Obviously ℓ is only invariant under the affine Weyl subgroup $\mathcal{W}(E_9)$.

To get a grip on the Lie algebra of physical states $\mathfrak{g}_\Lambda$ associated with the lattice $II_{9,1}$ we now employ the DDF construction well known from string theory (see, e.g., [17]). It provides a convenient way of obtaining all physical states by applying the DDF operators to the tachyonic ground states. A special feature of subcritical algebras, i.e. algebras with rank $d < 26$, is the relevance of *longitudinal* DDF operators for the spectrum of physical states.

Let us recall the basic features of the DDF construction. One starts with a tachyonic ground state of momentum $\boldsymbol{v}$ with $\boldsymbol{v}^2 = 2$ and an associated light-like vector $\boldsymbol{k}(\boldsymbol{v})$ with $\boldsymbol{k}^2 = 0$ and $\boldsymbol{v} \cdot \boldsymbol{k} = 1$. For continuous momenta, such vectors can always be found and rotated into a convenient frame, but on the lattice this is in general not possible. For this reason we have to work with the rationalization of the root lattice, that is $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$, to incorporate the intermediate points. To see this let us define the *DDF decomposition* of a given level-ℓ root $\boldsymbol{\Lambda}$ in the fundamental Weyl chamber by

$$\boldsymbol{\Lambda} = \boldsymbol{v} - n\boldsymbol{k} \tag{25}$$

where $\boldsymbol{k} \equiv \boldsymbol{k}(\boldsymbol{v}) := -\ell^{-1}\boldsymbol{\delta}$, and $n := 1 - \frac{1}{2}\boldsymbol{\Lambda}^2$ is the number of steps required to reach the root by starting from $\boldsymbol{v}$ and decreasing the momentum by $\boldsymbol{k}$ at each step. Notice that we have a factor of ℓ^{-1} in the definition of $\boldsymbol{k}$ at level ℓ and so beyond level one neither $\boldsymbol{k}$ nor $\boldsymbol{v}$ will be on the lattice in general.

In addition we need a set of transversal polarization vectors $\boldsymbol{\xi}_i = \boldsymbol{\xi}_i(\boldsymbol{v}, \boldsymbol{k})$ subject to $\boldsymbol{\xi}_i \cdot \boldsymbol{v} = \boldsymbol{\xi}_i \cdot \boldsymbol{k} = 0$, which for convenience we choose to be orthonormalized. We use the letters $i, j, \ldots = 1, \ldots, 8$ to label the transversal indices. Given these data, we can define the *transversal DDF operators* [6] by

$$A^i_m(\boldsymbol{v}) := \mathrm{Res}_z\left[\boldsymbol{\xi}_i(z)\mathcal{V}(\mathrm{e}^{m\boldsymbol{k}}, z)\right] \tag{26}$$

which describe the emission of a photon with momentum $m\boldsymbol{k}$ and polarization $\boldsymbol{\xi}_i$ from the string. Note that no normal-ordering is required here owing to the fact that $\boldsymbol{k}$ is light-like and is orthogonal to the polarization vectors. Normal-ordering is, however, required in the definition of the *longitudinal DDF operators* [7] given by

$$A^-_m(\boldsymbol{v}) := \mathcal{L}_m(\boldsymbol{v}) - \frac{1}{2}\sum_{i=1}^{8}\sum_{n\in\mathbb{Z}} {}^\times_\times A^i_n(\boldsymbol{v})A^i_{m-n}(\boldsymbol{v}){}^\times_\times + 2\delta_{m0}\boldsymbol{k}\cdot\boldsymbol{p} \tag{27}$$

with

$$\mathcal{L}_m(\boldsymbol{v}) := \mathrm{Res}_z\left[-\mathcal{V}(\boldsymbol{v}(-1)\,\mathrm{e}^{m\boldsymbol{k}}, z) + \frac{m}{2}(\boldsymbol{v}\cdot\boldsymbol{k})\frac{\mathrm{d}}{\mathrm{d}z}\log\left(\frac{\boldsymbol{k}(z)}{\boldsymbol{v}\cdot\boldsymbol{k}}\right)\mathcal{V}(\mathrm{e}^{m\boldsymbol{k}}, z)\right] \tag{28}$$

where we used $\boldsymbol{k}\cdot\boldsymbol{v} = 1$. The normal-ordering ${}^\times_\times \cdots {}^\times_\times$ places DDF operators with positive index to the left and operators with negative index to the right. The longitudinal operators A^-_m realize a Virasoro algebra with central charge $c = 26 - d$, the transversal operators realize a $(d-2)$-dimensional Heisenberg algebra, and both families of operators commute with each other. In the sequel we will be careful to indicate the dependence of the DDF operators on their tachyonic momenta since we will have to calculate commutators of DDF operators associated with *different* momenta.

Let us return to the hyperbolic algebra E_{10}. Any level-2 root in C must be of the form $\boldsymbol{\Lambda}_1 + n\boldsymbol{\delta}$ or $\boldsymbol{\Lambda}_7 + n\boldsymbol{\delta}$ or $2\boldsymbol{\Lambda}_0 + n\boldsymbol{\delta}$ for some $n \in \mathbb{N}$. In [8] the 192-dimensional root space of the root $\boldsymbol{\Lambda}_7$ was determined. We will now discuss the next non-trivial example, the root $\boldsymbol{\Lambda}_1$, dual to the simple root $\boldsymbol{r}_1$. Explicitly, $\boldsymbol{\Lambda}_1$ is given by

$$\boldsymbol{\Lambda}_1 = \left[\begin{array}{ccccccccc} & & & & & & 9 & & \\ 2 & 4 & 6 & 9 & 12 & 15 & 18 & 12 & 6 \end{array}\right] = (0,0,0,0,0,0,1,1,1|3) \tag{29}$$

hence $\mathbf{\Lambda}_1^2 = -6$. The dimension of the associated root space is 727. This deviates by one from the number of transversal states, which is 726. Since, on the other hand, it is well known that the level-1 states are precisely given by the transversal DDF states, one might therefore be tempted to conjecture that the 727-dimensional root space is spanned by the 726 transversal states and one distinguished longitudinal state. As we will see, however, this is not true and hints at some additional hidden structure inside E_{10}.

In general, one has the level-2 multiplicity formula [16]

$$\mathrm{mult}(\mathbf{\Lambda}) = \xi(3 - \tfrac{1}{2}\mathbf{\Lambda}^2) \tag{30}$$

where ξ is defined in terms of the generating function

$$\sum_{n\geqslant 0} \xi(n) q^n = \phi(q)^{-8}\left[1 - \phi(q^2)/\phi(q^4)\right] \qquad \phi(q) = \prod_{l=1}^{\infty}(1-q^l)$$

denoting the Euler function. Recall that we noticed earlier that the appearance of longitudinal states is generic for *any* level-2 root, and that therefore this deviation comes as no surprise, but from this example the emergence of the longitudinal states could have been anticipated. Our general DDF decomposition simplifies to

$$\mathbf{\Lambda}_1 = \boldsymbol{r} + \boldsymbol{s} + m\boldsymbol{\delta} \tag{31}$$

where $\boldsymbol{r}$ and $\boldsymbol{s}$ are two real positive level-1 roots. In general there will be many different ways to split $\mathbf{\Lambda}_1$ in this manner. These are all related by elements of the finite group $\mathcal{W}(\mathbf{\Lambda}_1)$, the stabilizer of $\mathbf{\Lambda}_1$, whereas the decompositions for fixed m are invariant under the little group $\mathcal{W}(\mathbf{\Lambda}_1, \boldsymbol{\delta})$, which is the (finite) subgroup of the full hyperbolic Weyl group leaving both $\mathbf{\Lambda}_1$ and $\boldsymbol{\delta}$ invariant. Thus, for every m, we will choose one of these decompositions, work out all commutators, and act on the resultant equations with the little Weyl group to obtain all possible states. For different values of m the situation is not that simple, owing to the fact that the little Weyl group is infinite dimensional in this case. Therefore we will deal with these decompositions case by case and combine the resulting states in the end.

We will need two different decompositions:

(i) $\mathbf{\Lambda}_1 = \boldsymbol{r} + \boldsymbol{s} + 3\boldsymbol{\delta}$ with

$$\begin{aligned}
\boldsymbol{r} &:= \begin{bmatrix} & & & & & & 0 & & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = (0,0,0,0,0,0,0,1,-1|0) \\
\boldsymbol{s} &:= \begin{bmatrix} & & & & & & 0 & & \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = (0,0,0,0,0,0,1,0,-1|0)
\end{aligned} \tag{32}$$

(ii) $\mathbf{\Lambda}_1 = \boldsymbol{r}' + \boldsymbol{s}' + 2\boldsymbol{\delta}$ with

$$\begin{aligned}
\boldsymbol{r}' &:= \begin{bmatrix} & & & & & & 0 & & \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = (0,0,0,0,0,1,0,0,-1|0) \\
\boldsymbol{s}' &:= \begin{bmatrix} & & & & & & 3 & & \\ 1 & 1 & 1 & 3 & 4 & 5 & 6 & 4 & 2 \end{bmatrix} = (0,0,0,0,0,-1,1,1,0|1).
\end{aligned} \tag{33}$$

Although we will need several sets of polarization vectors adjusted to the different DDF decompositions, we will present the basis using the following set, which is adjusted to the first decomposition (32):

$$\begin{aligned}
&\boldsymbol{\xi}_\alpha \equiv \boldsymbol{\xi}_\alpha(\boldsymbol{r}) = \boldsymbol{\xi}_\alpha(\boldsymbol{s}) = \boldsymbol{\xi}_\alpha(\boldsymbol{a}) \qquad \text{for } \alpha = 1, \ldots, 7 \\
&\boldsymbol{\xi}_1 = (1,0,0,0,0,0,0,0,0|0) \\
&\vdots
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\xi}_6 &= (0,0,0,0,0,1,0,0,0|0)\\
\boldsymbol{\xi}_7 &= \tfrac{1}{2}\sqrt{2}(0,0,0,0,0,0,1,1,1|1)\\
\boldsymbol{\xi}_8(\boldsymbol{a}) &= \tfrac{1}{2}\sqrt{2}(0,0,0,0,0,0,1,-1,0|0)\\
\boldsymbol{\xi}_8(\boldsymbol{r}) &= \tfrac{1}{2}\sqrt{2}(0,0,0,0,0,0,-1,1,1|1)\\
\boldsymbol{\xi}_8(\boldsymbol{s}) &= \tfrac{1}{2}\sqrt{2}(0,0,0,0,0,0,1,-1,1|1).
\end{aligned}
\tag{34}
$$

For the little group we have $\mathcal{W}(\boldsymbol{\Lambda}_1, \boldsymbol{\delta}) \equiv \mathbb{Z}_2 \times \mathcal{W}(E_7)$, the group generated by the eight fundamental reflections $\{w_0, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$. On the above system of polarization vectors the group acts as follows:

$$
\begin{aligned}
&w_0(\boldsymbol{\xi}_8) = -\boldsymbol{\xi}_8\\
&w_2(\boldsymbol{\xi}_5) = \boldsymbol{\xi}_6,\ w_2(\boldsymbol{\xi}_6) = \boldsymbol{\xi}_5\\
&w_3(\boldsymbol{\xi}_4) = \boldsymbol{\xi}_5,\ w_3(\boldsymbol{\xi}_5) = \boldsymbol{\xi}_4\\
&w_4(\boldsymbol{\xi}_3) = \boldsymbol{\xi}_4,\ w_4(\boldsymbol{\xi}_4) = \boldsymbol{\xi}_3\\
&w_5(\boldsymbol{\xi}_2) = \boldsymbol{\xi}_3,\ w_5(\boldsymbol{\xi}_3) = \boldsymbol{\xi}_2\\
&w_8(\boldsymbol{\xi}_1) = \boldsymbol{\xi}_2,\ w_8(\boldsymbol{\xi}_2) = \boldsymbol{\xi}_1\\
&w_6(\boldsymbol{\xi}_1) = -\boldsymbol{\xi}_2,\ w_6(\boldsymbol{\xi}_2) = -\boldsymbol{\xi}_1\\
&w_7(\boldsymbol{\xi}_i) = \boldsymbol{\xi}_i - \frac{1}{4}\sum_{j=1}^{6}\boldsymbol{\xi}_j - \frac{1}{4}\sqrt{2}\boldsymbol{\xi}_7 \qquad i = 1\ldots 6\\
&w_7(\boldsymbol{\xi}_7) = \frac{1}{2}\boldsymbol{\xi}_7 - \frac{1}{4}\sqrt{2}\sum_{j=1}^{6}\boldsymbol{\xi}_j.
\end{aligned}
\tag{35}
$$

The DDF construction provides us with an explicit basis for the physical states of momentum $\boldsymbol{\Lambda}_1$. We have 726 transversal and 54 longitudinal states,

$$
\begin{aligned}
&A^i_{-1}(\boldsymbol{a})A^j_{-1}(\boldsymbol{a})A^k_{-1}(\boldsymbol{a})A^l_{-1}(\boldsymbol{a})|\boldsymbol{a}\rangle\\
&A^i_{-1}(\boldsymbol{a})A^j_{-1}(\boldsymbol{a})A^k_{-2}(\boldsymbol{a})|\boldsymbol{a}\rangle\\
&A^i_{-2}(\boldsymbol{a})A^j_{-2}(\boldsymbol{a})|\boldsymbol{a}\rangle\\
&A^i_{-1}(\boldsymbol{a})A^j_{-3}(\boldsymbol{a})|\boldsymbol{a}\rangle\\
&A^i_{-4}(\boldsymbol{a})|\boldsymbol{a}\rangle\\
&A^i_{-1}(\boldsymbol{a})A^j_{-1}(\boldsymbol{a})A^-_{-2}(\boldsymbol{a})|\boldsymbol{a}\rangle\\
&A^i_{-2}(\boldsymbol{a})A^-_{-2}(\boldsymbol{a})|\boldsymbol{a}\rangle\\
&A^i_{-1}(\boldsymbol{a})A^-_{-3}(\boldsymbol{a})|\boldsymbol{a}\rangle\\
&A^-_{-2}(\boldsymbol{a})A^-_{-2}(\boldsymbol{a})|\boldsymbol{a}\rangle\\
&A^-_{-4}(\boldsymbol{a})|\boldsymbol{a}\rangle.
\end{aligned}
\tag{36}
$$

Therefore we can express any element of the root space $E_{10}^{(\boldsymbol{\Lambda}_1)}$ as a linear combination of these states. Since the E_{10} states are precisely the elements of $\mathfrak{g}_{II_{9,1}}^{(\boldsymbol{\Lambda}_1)}$ that are expressible by commutators of level-1 elements, the problem is to work out the 'Clebsch–Gordan coefficients' occurring in the expansion

$$
\begin{aligned}
&[A^{i_1}_{-m_1}(\boldsymbol{s})\ldots A^{i_M}_{-m_M}(\boldsymbol{s})|\boldsymbol{s}\rangle], A^{j_1}_{-n_1}(\boldsymbol{r})\ldots A^{j_N}_{-n_N}(\boldsymbol{r})|\boldsymbol{r}\rangle]\\
&\quad = \sum_{\substack{p_1+\cdots+q_Q=n\\ k_1,\ldots,k_P}} c^{i_1\ldots i_M j_1\ldots j_N}_{k_1\ldots k_P} A^{k_1}_{-p_1}(\boldsymbol{a})\ldots A^{k_P}_{-p_P}(\boldsymbol{a})A^-_{-q_1}(\boldsymbol{a})\ldots A^-_{-q_Q}(\boldsymbol{a})|\boldsymbol{a}\rangle
\end{aligned}
$$

into which all the information on how E_{10} sits inside the Lie algebra of physical states is encoded. We found that the following 727 states form a complete basis of the root space $E_{10}^{(\Lambda_1)}$:

$$A^i_{-4}|\boldsymbol{a}\rangle$$
$$A^i_{-2}A^j_{-2}|\boldsymbol{a}\rangle$$
$$A^i_{-2}A^-_{-2}|\boldsymbol{a}\rangle$$
$$A^-_{-2}A^-_{-2}|\boldsymbol{a}\rangle$$
$$A^\alpha_{-1}A^\beta_{-1}A^\gamma_{-1}A^7_{-1}|\boldsymbol{a}\rangle$$
$$\{A^\mu_{-1}A^\nu_{-3} - A^\mu_{-3}A^\nu_{-1}\}|\boldsymbol{a}\rangle$$
$$\{A^8_{-1}A^\mu_{-3} + 3A^\mu_{-1}A^8_{-3}\}|\boldsymbol{a}\rangle$$
$$\{A^i_{-1}A^i_{-1}A^i_{-1}A^i_{-1} - 2A^i_{-1}A^i_{-3}\}|\boldsymbol{a}\rangle^{(1)}$$
$$\{A^\alpha_{-1}A^\alpha_{-1}A^\alpha_{-1}A^\beta_{-1} - A^\alpha_{-3}A^\beta_{-1}\}|\boldsymbol{a}\rangle$$
$$\{A^\alpha_{-1}A^\alpha_{-1}A^\alpha_{-1}A^7_{-1} + 5A^\alpha_{-3}A^7_{-1}\}|\boldsymbol{a}\rangle$$
$$\{A^\alpha_{-1}A^7_{-1}A^7_{-1}A^7_{-1} + A^\alpha_{-3}A^7_{-1}\}|\boldsymbol{a}\rangle$$
$$\{A^\mu_{-1}A^8_{-1}A^8_{-1}A^8_{-1} + A^\mu_{-1}A^8_{-3}\}|\boldsymbol{a}\rangle$$
$$\{A^\alpha_{-1}A^\beta_{-1}A^\gamma_{-1}A^\gamma_{-1} + A^\alpha_{-3}A^\beta_{-1}\}|\boldsymbol{a}\rangle$$
$$\{A^\alpha_{-1}A^7_{-1}A^\beta_{-1}A^\beta_{-1} - A^\alpha_{-3}A^7_{-1}\}|\boldsymbol{a}\rangle$$
$$\{A^\alpha_{-1}A^\beta_{-1}A^7_{-1}A^7_{-1} - A^\alpha_{-3}A^\beta_{-1}\}|\boldsymbol{a}\rangle$$
$$\{A^\alpha_{-1}A^\beta_{-1}A^\gamma_{-1}A^\delta_{-1} - 2A^\epsilon_{-3}A^\eta_{-1}\}|\boldsymbol{a}\rangle$$
$$\left\{\left(A^i_{-1}A^j_{-1} + \delta^{ij}\frac{1}{3}A^8_{-1}A^8_{-1}\right)A^l_{-2}\right\}|\boldsymbol{a}\rangle^{(2)}$$
$$\left\{\left(A^i_{-1}A^j_{-1} + \delta^{ij}\frac{1}{3}A^8_{-1}A^8_{-1}\right)A^-_{-2}\right\}|\boldsymbol{a}\rangle^{(2)}$$
$$\left\{A^8_{-1}A^8_{-1}A^-_{-2} + \frac{3}{2}\sum_{i=1}^{8}A^i_{-1}A^i_{-3}\right\}|\boldsymbol{a}\rangle$$
$$\left\{A^\alpha_{-1}A^\alpha_{-1}A^\alpha_{-1}A^\alpha_{-1} + 2A^\alpha_{-1}A^\alpha_{-3} + 3A^7_{-1}A^7_{-3} - \sum_{\rho=1}^{7}A^\rho_{-1}A^\rho_{-3}\right\}|\boldsymbol{a}\rangle$$
$$\left\{A^\alpha_{-1}A^\alpha_{-1}A^7_{-1}A^7_{-1} + 3A^\alpha_{-1}A^\alpha_{-3} + A^7_{-1}A^7_{-3} - \frac{2}{3}\sum_{\rho=1}^{7}A^\rho_{-1}A^\rho_{-3}\right\}|\boldsymbol{a}\rangle$$
$$\left\{A^\mu_{-1}A^\nu_{-1}A^8_{-1}A^8_{-1} - A^\mu_{-1}A^\nu_{-3} + \delta^{\mu\nu}\left(A^8_{-1}A^8_{-3} + \frac{2}{3}\sum_{\rho=1}^{7}A^\rho_{-1}A^\rho_{-3}\right)\right\}|\boldsymbol{a}\rangle$$
$$\left\{A^\mu_{-1}A^\nu_{-1}A^\sigma_{-1}A^8_{-1} + \frac{1}{3}\delta^{\mu\nu}A^8_{-1}A^\sigma_{-3} + \frac{1}{3}\delta^{\nu\sigma}A^8_{-1}A^\mu_{-3} + \frac{1}{3}\delta^{\mu\sigma}A^8_{-1}A^\nu_{-3}\right\}|\boldsymbol{a}\rangle$$
$$\left\{A^\alpha_{-1}A^\alpha_{-1}A^\beta_{-1}A^\beta_{-1} - A^\alpha_{-1}A^\alpha_{-3} - A^\beta_{-1}A^\beta_{-3} - A^7_{-1}A^7_{-3} + \frac{1}{3}\sum_{\mu=1}^{7}A^\mu_{-1}A^\mu_{-3}\right\}|\boldsymbol{a}\rangle.$$

Here we use the following conventions: roman letters $(i, j, \ldots)$ run from one to eight, greek letters from the middle of the alphabet $(\mu, \nu, \ldots)$ run from one to seven and greek letters from the beginning of the alphabet $(\alpha, \beta, \ldots)$ run from one to six, with the exceptions $^{(1)}i \in \{7, 8\}$ and $^{(2)}(i, j) \neq (8, 8)$.

As can be seen, the calculated states fall into two classes, depending on whether the indices of the occurring DDF operators are odd or even. Recall that the DDF construction of level-2 states involves intermediate states with momenta of the type $\frac{1}{2}m\delta$. Therefore, precisely the operators with odd indices generate states that do not lie on the lattice and it can be seen that these are responsible for the more complicated basis elements.

In view of current work [18] on the structure of the space of missing states let us present a basis for the orthogonal complement of $E_{10}^{(\mathbf{\Lambda}_1)}$ within the full space of physical states. A direct calculation using the string scalar product reveals that the 53-dimensional orthogonal complement of $E_{10}^{(\mathbf{\Lambda}_1)}$ in $\mathfrak{g}_{II_{9,1}}^{(\mathbf{\Lambda}_1)}$ is spanned by the following states:

$$A^i_{-1}A^-_{-3}|\boldsymbol{a}\rangle$$

$$\left\{A^8_{-1}A^8_{-1}A^i_{-2} - \frac{1}{3}\sum_{\mu=1}^{7} A^\mu_{-1}A^\mu_{-1}A^i_{-2}\right\}|\boldsymbol{a}\rangle$$

$$\{A^-_{-2}A^-_{-2} - 4A^-_{-4}\}|\boldsymbol{a}\rangle$$

$$\left\{A^\mu_{-1}A^8_{-1}A^8_{-1}A^8_{-1} - 3\sum_{\nu=1}^{7} A^\nu_{-1}A^\nu_{-1}A^\mu_{-1}A^8_{-1} - 2A^\mu_{-1}A^8_{-3} + 6A^8_{-1}A^\mu_{-3}\right\}|\boldsymbol{a}\rangle$$

$$\left\{A^\alpha_{-1}A^7_{-3} + A^7_{-1}A^\alpha_{-3} - 2A^\alpha_{-1}A^7_{-1}A^7_{-1}A^7_{-1} - 4A^\alpha_{-1}A^\alpha_{-1}A^\alpha_{-1}A^7_{-1} + \frac{3}{2}\sum_{i=1}^{8} A^i_{-1}A^i_{-1}A^\alpha_{-1}A^7_{-1}\right\}|\boldsymbol{a}\rangle$$

$$\left\{A^\alpha_{-3}A^\alpha_{-1} - \frac{3}{2}A^8_{-3}A^8_{-1} + \frac{1}{2}A^7_{-3}A^7_{-1} - A^\alpha_{-1}A^\alpha_{-1}A^\alpha_{-1}A^\alpha_{-1} - \frac{1}{4}A^7_{-1}A^7_{-1}A^7_{-1}A^7_{-1} + \frac{3}{4}\sum_{i=1}^{8} A^i_{-1}A^i_{-1}A^\alpha_{-1}A^\alpha_{-1} + \frac{3}{8}\sum_{i=1}^{8} A^i_{-1}A^i_{-1}A^7_{-1}A^7_{-1} - A^\alpha_{-1}A^\alpha_{-1}A^7_{-1}A^7_{-1} + \frac{3}{8}\sum_{\mu=1}^{7} A^\mu_{-1}A^\mu_{-1}A^8_{-1}A^8_{-1} - \frac{3}{8}A^8_{-1}A^8_{-1}A^8_{-1}A^8_{-1}\right\}|\boldsymbol{a}\rangle$$

$$\left\{A^\alpha_{-1}A^\beta_{-3} + A^\beta_{-1}A^\alpha_{-3} + \frac{1}{2}A^\alpha_{-1}A^\beta_{-1}A^\beta_{-1}A^\beta_{-1} + \frac{1}{2}A^\alpha_{-1}A^\alpha_{-1}A^\alpha_{-1}A^\beta_{-1} - \frac{3}{2}\sum_{\substack{\gamma=1\\ \gamma\neq\alpha,\beta}}^{6} A^\alpha_{-1}A^\beta_{-1}A^\gamma_{-1}A^\gamma_{-1} + \frac{3}{2}A^\alpha_{-1}A^\beta_{-1}A^8_{-1}A^8_{-1} + \frac{3}{2}A^\alpha_{-1}A^\beta_{-1}A^7_{-1}A^7_{-1} + 4A^\gamma_{-1}A^\delta_{-1}A^\epsilon_{-1}A^\eta_{-1}\right\}|\boldsymbol{a}\rangle$$

$$\left\{\frac{4}{3}A^8_{-1}A^8_{-3} - \frac{1}{8}\sum_{\mu=1}^{7} A^\mu_{-1}A^\mu_{-1}A^-_{-2} + \frac{3}{8}A^8_{-1}A^8_{-1}A^-_{-2} + \frac{1}{3}A^8_{-1}A^8_{-1}A^8_{-1}A^8_{-1} + \frac{1}{4}\sum_{\mu=1}^{7} A^\mu_{-1}A^\mu_{-1}A^8_{-1}A^8_{-1}\right\}|\boldsymbol{a}\rangle$$

$$\left\{7A^8_{-1}A^8_{-3} + \frac{7}{4}A^8_{-1}A^8_{-1}A^8_{-1}A^8_{-1} - \frac{5}{2}\sum_{\mu=1}^{7} A^\mu_{-1}A^\mu_{-1}A^8_{-1}A^8_{-1} - \frac{1}{4}\sum_{\mu,\nu=1}^{7} A^\mu_{-1}A^\mu_{-1}A^\nu_{-1}A^\nu_{-1} - \sum_{\mu=1}^{7} A^\mu_{-1}A^\mu_{-3}\right\}|\boldsymbol{a}\rangle.$$

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Appendix A. Calculations

The calculations where performed using a Lisp program and the symbolic algebra system MAPLE.

We will first consider the decomposition of $\mathbf{\Lambda}_1$ with $m = 3$. Since the calculations for the other decomposition are similar we will treat this example in detail and skip the other case. One starts by collecting all level-1 roots that upon commutation give states in this particular root space. Since the DDF construction provides us with explicit expressions for all level-1 roots this is easily done. We arrive at the following list of commutators:

$$\begin{array}{ll} [|\boldsymbol{s}\rangle, A^i_{-3}|\boldsymbol{r}\rangle] & [A^i_{-1}|\boldsymbol{s}\rangle, A^j_{-2}|\boldsymbol{r}\rangle] \\ [|\boldsymbol{s}\rangle, A^i_{-2}A^j_{-1}|\boldsymbol{r}\rangle] & [A^k_{-1}|\boldsymbol{s}\rangle, A^i_{-1}A^j_{-1}|\boldsymbol{r}\rangle] \\ [|\boldsymbol{s}\rangle, A^i_{-1}A^j_{-1}A^k_{-1}|\boldsymbol{r}\rangle] & \end{array} \tag{A1}$$

where the indices i, j, k run from 1 to 8. We will explain the evaluation of commutators of this type by working out one example in detail, for which we take the commutator $[|\boldsymbol{s}\rangle, A^\alpha_{-2}A^8_{-1}|\boldsymbol{r}\rangle]$.

The main technical problem for the calculations is that we have to deal with two different objects. We started with the set of oscillators $\{\alpha^\mu_m, \mu = 1 \ldots d, m \in \mathbb{Z}\}$ acting on our Fock space $\mathcal{F}$. In terms of these we defined the vertex operators and only here can we perform the explicit calculations. We then restricted ourselves to the subspace of physical states $\mathcal{P}^1$ and introduced the DDF operators providing us with a elegant way to obtain explicit bases for the subspaces of $\mathcal{P}^1$ of some definite momentum. The price we have to pay for this elegance is that it is not possible to perform explicit calculations using the DDF operators.

Returning to our example, we start by determining the corresponding oscillator representations of the commuted states. We find

$$\begin{aligned} A^\alpha_{-2}A^8_{-1}|\boldsymbol{r}\rangle &= [\boldsymbol{\xi}_\alpha(-1)|2\delta\rangle, [\boldsymbol{\xi}_8(-1)|\boldsymbol{\delta}\rangle, |\boldsymbol{r}\rangle]] \\ &= \epsilon(\boldsymbol{k}, \boldsymbol{r})\left\{\boldsymbol{\xi}_\alpha(-2)\boldsymbol{\xi}_8(-1) + 2\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\xi}_8(-1)\boldsymbol{\delta}(-1)\right\}|\boldsymbol{r} + 3\boldsymbol{\delta}\rangle \end{aligned} \tag{A2}$$

where $\epsilon(\boldsymbol{k}, \boldsymbol{r})$ denotes the cocycle factor owing to the twisted group algebra $\mathbb{R}\{II_{9,1}\}$. For the calculation of the commutator we need the vertex operator corresponding to $|\boldsymbol{r}\rangle$ which is given by the definition (14)

$$\mathcal{V}(|\boldsymbol{r}\rangle, z) = \exp\left(\int \boldsymbol{r}_<(z)\,\mathrm{d}z\right)\mathrm{e}^{\boldsymbol{r}} z^{\boldsymbol{r}(0)} \exp\left(\int \boldsymbol{r}_>(z)\,\mathrm{d}z\right).$$

Now we can calculate the commutator, for which we find the expression

$$\begin{aligned} &[|\boldsymbol{s}\rangle, A^\alpha_{-2}A^8_{-1}|\boldsymbol{r}\rangle] \\ &\quad = \epsilon\{-\tfrac{4}{3}\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\xi}_8(-3) - \tfrac{1}{2}\boldsymbol{\xi}_8(-2)\boldsymbol{\xi}_\alpha(-2) - \tfrac{1}{3}\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\xi}_8(-1)^3 \\ &\qquad -\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\delta}(-1)\boldsymbol{\xi}_8(-2) - \tfrac{1}{2}\boldsymbol{\xi}_8(-1)\boldsymbol{\delta}(-1)\boldsymbol{\xi}_\alpha(-2) \end{aligned}$$

$$
\begin{aligned}
&-\tfrac{1}{2}\boldsymbol{\xi}_8(-1)\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\delta}(-1)^2-\tfrac{1}{2}\boldsymbol{\xi}_8(-2)\boldsymbol{\Lambda}(-1)\boldsymbol{\xi}_\alpha(-1)-\tfrac{4}{3}\sqrt{2}\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\delta}(-3)\\
&+\tfrac{1}{8}\sqrt{2}\boldsymbol{\xi}_\alpha(-2)\boldsymbol{\Lambda}(-1)^2+\tfrac{1}{24}\sqrt{2}\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\Lambda}(-1)^3+\tfrac{3}{8}\boldsymbol{\xi}_\alpha(-2)\boldsymbol{\delta}(-1)^2\sqrt{2}\\
&-\tfrac{1}{4}\boldsymbol{\xi}_\alpha(-2)\sqrt{2}\boldsymbol{\xi}_8(-1)^2+\tfrac{7}{12}\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\delta}(-1)^3\sqrt{2}\\
&+\tfrac{1}{4}\sqrt{2}\boldsymbol{\xi}_\alpha(-2)\boldsymbol{\Lambda}(-2)-\tfrac{3}{4}\sqrt{2}\boldsymbol{\xi}_\alpha(-2)\boldsymbol{\delta}(-2)+\tfrac{1}{3}\sqrt{2}\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\Lambda}(-3)\\
&-\boldsymbol{\xi}_8(-1)\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\delta}(-2)-\boldsymbol{\xi}_\alpha(-1)\sqrt{2}\boldsymbol{\xi}_8(-1)\boldsymbol{\xi}_8(-2)\\
&-\tfrac{1}{2}\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\delta}(-1)\sqrt{2}\boldsymbol{\xi}_8(-1)^2-\tfrac{1}{4}\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\Lambda}(-1)\sqrt{2}\boldsymbol{\xi}_8(-1)^2\\
&-\tfrac{1}{2}\boldsymbol{\xi}_\alpha(-2)\boldsymbol{\delta}(-1)\boldsymbol{\Lambda}(-1)\sqrt{2}-\tfrac{5}{8}\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\delta}(-1)^2\boldsymbol{\Lambda}(-1)\sqrt{2}\\
&+\tfrac{1}{4}\sqrt{2}\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\Lambda}(-2)\boldsymbol{\Lambda}(-1)-\tfrac{3}{4}\sqrt{2}\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\Lambda}(-1)\boldsymbol{\delta}(-2)\\
&-\tfrac{1}{2}\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\xi}_8(-1)\boldsymbol{\delta}(-1)\boldsymbol{\Lambda}(-1)-\tfrac{1}{2}\sqrt{2}\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\delta}(-1)\boldsymbol{\delta}(-2)\}|\boldsymbol{a}\rangle \qquad \text{(A3)}
\end{aligned}
$$

where $\epsilon \equiv \epsilon(\boldsymbol{s}-\boldsymbol{k},\boldsymbol{r})$. To compare this result with the basis of the subspace of physical states we have to re-express the oscillators in terms of DDF operators. Here we encounter another subtlety; we used the general Fock space for calculations where elements of the Lie algebra of physical states are defined only modulo null states, that is, states of the form $\mathrm{L}_{(-1)}\psi$ for a general state ψ. To obtain equivalence and to explicitly solve the system of linear equations we have to include a general null state. We arrive at

$$
\begin{aligned}
&[|\boldsymbol{s}\rangle, A^\alpha_{-2}A^8_{-1}|\boldsymbol{r}\rangle]\\
&\quad=\epsilon\{-\tfrac{1}{2}\sqrt{2}A^\alpha_{-4}-\tfrac{1}{2}\sqrt{2}A^\alpha_{-1}A^8_{-1}A^8_{-2}+\tfrac{1}{3}A^\alpha_{-1}A^8_{-1}A^8_{-1}A^8_{-1}+\tfrac{1}{3}A^\alpha_{-1}A^8_{-3}\}|\boldsymbol{a}\rangle\\
&\qquad+\mathrm{L}_{(-1)}\{\tfrac{5}{8}\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\delta}(-1)^2\sqrt{2}-\tfrac{1}{24}\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\Lambda}(-1)^2\sqrt{2}+\tfrac{1}{4}\boldsymbol{\xi}_\alpha(-1)\sqrt{2}\boldsymbol{\xi}_8(-1)^2\\
&\qquad-\tfrac{1}{2}\boldsymbol{\xi}_8(-2)\,\boldsymbol{\xi}_\alpha(-1)-\tfrac{1}{12}\boldsymbol{\xi}_\alpha(-2)\,\boldsymbol{\Lambda}\,(-1)\sqrt{2}+\tfrac{1}{2}\boldsymbol{\delta}(-1)\,\boldsymbol{\xi}_\alpha(-2)\sqrt{2}\\
&\qquad+\tfrac{1}{6}\boldsymbol{\xi}_\alpha(-3)\sqrt{2}+\tfrac{3}{4}\boldsymbol{\xi}_\alpha(-1)\sqrt{2}-\tfrac{1}{6}\boldsymbol{\xi}_\alpha(-1)\sqrt{2}\boldsymbol{\Lambda}(-2)\boldsymbol{\delta}(-2)\\
&\qquad-\tfrac{1}{2}\boldsymbol{\xi}_8(-1)\boldsymbol{\delta}(-1)\boldsymbol{\xi}_\alpha(-1)\}|\boldsymbol{\Lambda}_1\rangle. \qquad \text{(A4)}
\end{aligned}
$$

Without calculation we record another commutator

$$
[A^8_{-1}|\boldsymbol{s}\rangle, A^\alpha_{-2}|\boldsymbol{r}\rangle]=\epsilon\{\tfrac{1}{2}\sqrt{2}A^\alpha_{-4}-\tfrac{1}{2}\sqrt{2}A^\alpha_{-1}A^8_{-1}A^8_{-2}-\tfrac{1}{3}A^\alpha_{-1}A^8_{-3}-\tfrac{1}{3}A^\alpha_{-1}A^8_{-1}A^8_{-1}A^8_{-1}\}|\boldsymbol{a}\rangle \qquad \text{(A5)}
$$

where we suppressed the $\mathrm{L}_{(-1)}\{\cdots\}$-term, as we will do from now on. To simplify these elements of E_{10} we take suitable linear combinations. We immediately get

$$
-\frac{1}{2\epsilon}\sqrt{2}[(A5)+(A4)]=A^\alpha_{-1}A^8_{-1}A^8_{-2}
$$

as our first element of a basis for $E_{10}^{(\boldsymbol{\Lambda}_1)}$. Another possibility is acting on equations like (A4) with the little Weyl group. Recalling that w_0 leaves $\boldsymbol{\xi}_1,\ldots\boldsymbol{\xi}_7$ invariant and changes the sign of $\boldsymbol{\xi}_8$ we have:

$$
\frac{1}{2\epsilon}\sqrt{2}[(A5)-w_0(A4)]=A^\alpha_{-4}.
$$

Evaluating all commutators and simplifying the resulting equation leads to a basis for a

516-dimensional subspace:

$$
\begin{gathered}
A^i_{-1}A^j_{-1}A^l_{-2}|\boldsymbol{a}\rangle^{(1)}\\
A^i_{-2}A^j_{-2}|\boldsymbol{a}\rangle\\
A^i_{-2}A^-_{-2}|\boldsymbol{a}\rangle\\
A^i_{-4}|\boldsymbol{a}\rangle\\
\{A^8_{-1}A^8_{-1}A^8_{-1}A^8_{-1} - 2A^8_{-1}A^8_{-3}\}|\boldsymbol{a}\rangle\\
\{A^\mu_{-1}A^8_{-1}A^8_{-1}A^8_{-1} + A^\mu_{-1}A^8_{-3}\}|\boldsymbol{a}\rangle\\
\{A^\mu_{-1}A^\nu_{-3} - A^\mu_{-3}A^\nu_{-1}\}|\boldsymbol{a}\rangle\\
\{3A^\mu_{-1}A^8_{-3} + A^8_{-1}A^\mu_{-3}\}|\boldsymbol{a}\rangle\\
\{A^\mu_{-1}A^8_{-1}A^-_{-2}\}|\boldsymbol{a}\rangle\\
\left\{A^\mu_{-1}A^\mu_{-1}A^i_{-1} + \frac{1}{3}A^8_{-1}A^8_{-1}A^i_{-2}\right\}|\boldsymbol{a}\rangle\\
\left\{A^\beta_{-1}A^\alpha_{-3} - A^\alpha_{-1}A^\beta_{-1}A^8_{-1}A^8_{-1} + \delta^{\alpha\beta}\left(\frac{1}{5}A^8_{-1}A^8_{-1}A^-_{-2} + \frac{1}{10}\sum_{\mu=1}^{7}A^\mu_{-1}A^\mu_{-1}A^8_{-1}A^8_{-1}\right)\right\}|\boldsymbol{a}\rangle\\
\left\{A^\alpha_{-1}A^\beta_{-1}A^-_{-1} + \frac{1}{2}A^\alpha_{-1}A^\beta_{-1}A^8_{-1}A^8_{-1} + \frac{1}{2}\sum_{\mu=1}^{7}A^\mu_{-1}A^\mu_{-1}A^\alpha_{-1}A^\beta_{-1} + \delta^{\alpha\beta}\left(A^8_{-1}A^8_{-3}\right.\right.\\
\left.\left.+\frac{1}{10}\sum_{\mu=1}^{7}A^\mu_{-1}A^\mu_{-1}A^8_{-1}A^8_{-1} + \frac{1}{5}A^8_{-1}A^8_{-1}A^-_{-2}\right)\right\}|\boldsymbol{a}\rangle\\
\left\{A^\alpha_{-1}A^\beta_{-1}A^\gamma_{-1}A^8_{-1} + \frac{1}{3}\delta^{\alpha\beta}A^8_{-1}A^\gamma_{-3} + \frac{1}{3}\delta^{\beta\gamma}A^8_{-1}A^\alpha_{-3} + \frac{1}{3}\delta^{\alpha\gamma}A^8_{-1}A^\beta_{-3}\right\}|\boldsymbol{a}\rangle\\
\left\{A^-_{-2}A^-_{-2} - A^8_{-1}A^8_{-1}A^-_{-2} - \frac{1}{4}\sum_{\mu,\nu=1}^{7}A^\mu_{-1}A^\mu_{-1}A^\nu_{-1}A^\nu_{-1} - \frac{3}{2}A^8_{-1}A^8_{-3}\right\}|\boldsymbol{a}\rangle.
\end{gathered}
$$

We use the following conventions: roman letters $(i, j, \ldots)$ run from one to eight and greek letters from the middle of the alphabet $(\mu, \nu, \ldots)$ run from one to seven, and we have the exception $^{(1)}i \neq j$.

For the decomposition with $m = 2$ we have to evaluate the commutators

$$
\begin{aligned}
&[|\boldsymbol{s}'\rangle, A^i_{-1}A^j_{-1}|\boldsymbol{r}'\rangle]\\
&[A^i_{-1}|\boldsymbol{s}'\rangle, A^j_{-1}|\boldsymbol{r}'\rangle].
\end{aligned}
\tag{A6}
$$

Analogous calculations here and combination of the two results lead to the basis given in the main part.

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